Instantaneous power $p(t) = i(t)v(t) = \frac{v^2(t)}{R} = i^2(t)R$

when $R=1$, $p(t) = v^2(t) = i^2(t)$

So, don't have to care whether our signal $g(t)$ is $v(t)$ or $i(t)$.

Calculation of the signals?

4.2 Energy and Power

Definition 4.12. For a signal $g(t)$, the instantaneous power $p(t)$ dissipated in the 1-Ω resister is $p_g(t) = |g(t)|^2$ regardless of whether $g(t)$ represents a voltage or a current. To emphasize the fact that this power is based upon unity resistance, it is often referred to as the **normalized (instantaneous) power**.

Definition 4.13. The total (normalized) ***energy*** of a signal $g(t)$ is given by

Let $y(t) = c\,g(t)$

$E_y = |c|^2 E_g$

$$E_g = \int_{-\infty}^{+\infty} p_g(t)dt = \int_{-\infty}^{+\infty} |g(t)|^2 dt = \lim_{T\to\infty} \int_{-T}^{T} |g(t)|^2 dt.$$

energy within time $\pm T$

4.14. By the **Parseval's theorem** discussed in 2.43, we have

$P_y \equiv \langle |y(t)|^2 \rangle = \langle |c\,g(t)|^2 \rangle = |c|^2 \langle |g(t)|^2 \rangle = |c|^2 P_g$

$$E_g = \int_{-\infty}^{\infty} |g(t)|^2 dt = \int_{-\infty}^{\infty} |G(f)|^2 df.$$

ESD: Energy spectral density

Definition 4.15. The average (normalized) ***power*** of a signal $g(t)$ is given by

"general formula"

$$P_g = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} |g(t)|^2 dt = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |g(t)|^2 dt. = \langle |g|^2 \rangle$$

energy

per unit time

Definition 4.16. To simplify the notation, there are two operators that used angle brackets to define two frequently-used integrals:

Important properties

① $\langle c \rangle = c$

② $\langle a g_1 + b g_2 \rangle = a\langle g_1 \rangle + b \langle g_2 \rangle$

(a) The "**time-average**" operator:

$$\langle g \rangle \equiv \langle g(t) \rangle \equiv \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} g(t)dt = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} g(t)dt \tag{42}$$

(b) The **inner-product** operator:

$$\langle g_1, g_2 \rangle \equiv \langle g_1(t), g_2(t) \rangle = \int_{-\infty}^{\infty} g_1(t) g_2^*(t) dt \tag{43}$$

4.17. Using the above definition, we may write

- $E_g = \langle g, g \rangle = \langle G, G \rangle$ where $G = \mathcal{F}\{g\}$
- $P_g = \left\langle |g|^2 \right\rangle$

Given a collection of numbers...

$5, -2, 1, 3, 4$

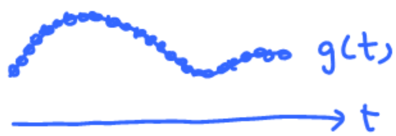
To find the "average", we compute

$$\frac{5+(-2)+(1)+3+4}{5}$$

add everything

total #

Finding the time-average of a func...



"add" everything

$$\langle g(t) \rangle \text{ ``=''} \frac{\displaystyle\int_{-\infty}^{\infty} g(t)\,dt}{\text{length of the interval } (-\infty,\infty)} = \lim_{T\to\infty} \frac{\displaystyle\int_{-T}^{T} g(t)\,dt}{2T}$$

area under the graph

total time

- Parseval's theorem: $\langle g_1, g_2 \rangle = \langle G_1, G_2 \rangle$
  where $G_1 = \mathcal{F}\{g_1\}$ and $G_2 = \mathcal{F}\{g_2\}$

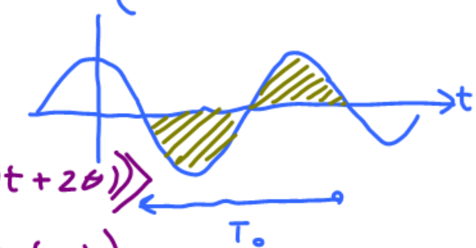
4.18. Time-Averaging over Periodic Signal: For ***periodic*** signal $g(t)$ with period T_0, the time-average operation in (42) can be simplified to

$$\langle g \rangle = \frac{1}{T_0} \int_{T_0} g(t)dt$$

where the integration is performed over a period of g.

Example 4.19. $\langle \cos(2\pi f_0 t + \theta) \rangle = \frac{1}{T_0}\int_{T_0} \cos(2\pi f_0 t + \theta)dt = \begin{cases} \cos\theta, & f_0 = 0 \\ 0, & f_0 \neq 0 \end{cases}$

Period $T_0 = \frac{1}{f_0}$



Similarly, $\langle \sin(2\pi f_0 t + \theta) \rangle = \begin{cases} \sin\theta, & f_0 = 0, \\ 0, & f_0 \neq 0 \end{cases}$

Example 4.20. $\langle \cos^2(2\pi f_0 t + \theta) \rangle = \langle \frac{1}{2}(1 + \cos(2\pi(2f_0)t + 2\theta)) \rangle$

When $f_0 = 0$, $= \cos^2\theta$

when $f_0 \neq 0$ $= \frac{1}{2}(1 + \underbrace{\cos(\quad)}_{\langle\ \rangle = 0}) = \frac{1}{2}$

Example 4.21. $\left\langle e^{j(2\pi f_0 t + \theta)} \right\rangle = \langle \cos(2\pi f_0 t + \theta) + j\sin(2\pi f_0 t + \theta) \rangle$

$= \begin{cases} e^{j\theta}, & f_0 = 0 \\ 0, & f_0 \neq 0 \end{cases}$

Example 4.22. Suppose $g(t) = ce^{j2\pi f_0 t}$ for some (possibly complex-valued) constant c and (real-valued) frequency f_0. Find P_g.

$P_g \equiv \langle |g(t)|^2 \rangle = \langle |c|^2 \underbrace{|e^{j2\pi f_0 t}|^2}_{1} \rangle = \langle |c|^2 \rangle = |c|^2$

4.23. When the signal $g(t)$ can be expressed in the form $g(t) = \sum_k c_k e^{j2\pi f_k t}$ and the f_k are distinct, then its (average) power can be calculated from

$$P_g = \sum_k |c_k|^2$$

very important assumption

$f_1 = 3$ $\quad$ $f_2 = 4$ $\quad$ $f_1 \neq f_2$

Example 4.24. Suppose $g(t) = 2e^{j6\pi t} + 3e^{j8\pi t}$. Find P_g.

$c_1 = 2$ $\quad$ $c_2 = 3$

$P_g = |c_1|^2 + |c_2|^2 = 2^2 + 3^2 = 4 + 9 = 13$

Example 4.25. Suppose $g(t) = 2e^{j6\pi t} + 3e^{j6\pi t}$. Find P_g.

$g(t) = 5e^{j6\pi t}$ $\quad$ $f_1 = 3$ $\quad$ $f_2 = 3$ $\quad$ $f_1 = f_2$

$P_g = 5^2 = 25$

Example 4.26. Suppose $g(t) = \cos(2\pi f_0 t + \theta)$. Find P_g.

Here, there are several ways to calculate P_g. We can simply use Example 4.20. Alternatively, we can first decompose the cosine into complex exponential functions using the Euler's formula:

$f_0 = 0$: $\cos\theta \Rightarrow P_g = \langle|\cos\theta|^2\rangle = \cos^2\theta$

$\cos(2\pi f_0 t + \theta) =$

$f_0 \neq 0$: $\frac{1}{2}\left(e^{j(2\pi f_0 t + \theta)} + e^{-j(2\pi f_0 t + \theta)}\right) = \underbrace{\tfrac{1}{2}e^{j\theta}}_{c_1} e^{j2\pi \underset{f_1}{f_0} t} + \underbrace{\tfrac{1}{2}e^{-j\theta}}_{c_2} e^{j2\pi \underset{f_2}{(-f_0)} t}$

$f_1 \neq f_2$ $\quad$ $P_g = |c_1|^2 + |c_2|^2 = \left(\tfrac{1}{2}\right)^2 + \left(\tfrac{1}{2}\right)^2 = \tfrac{1}{2}$

4.27. The (average) power of a sinusoidal signal $g(t) = A\cos(2\pi f_0 t + \theta)$ is

$|A| = \sqrt{2P_g}$

$$P_g = \begin{cases} \frac{1}{2}|A|^2, & f_0 \neq 0, \\ |A|^2 \cos^2\theta, & f_0 = 0. \end{cases}$$

This property means any sinusoid with nonzero frequency can be written in the form

$$g(t) = \sqrt{2P_g}\cos(2\pi f_0 t + \theta).$$

4.28. Extension of 4.27: Consider sinusoids $A_k \cos(2\pi f_k t + \theta_k)$ whose frequencies are positive and distinct. The (average) power of their sum

$$g(t) = \sum_k A_k \cos(2\pi f_k t + \theta_k)$$

is

$$P_g = \frac{1}{2}\sum_k |A_k|^2.$$

A_1 $\quad f_1 = \sqrt{3}$ $\quad A_2$ $\quad f_2 = \sqrt{5}$

Example 4.29. Suppose $g(t) = 2\cos\left(2\pi\sqrt{3}t\right) + 4\cos\left(2\pi\sqrt{5}t\right)$. Find P_g.

$$0 < f_1 \neq f_2 > 0 \Rightarrow P_g = \frac{1}{2}|A_1|^2 + \frac{1}{2}|A_2|^2 = \frac{1}{2}\left(2^2+4^2\right) = \frac{1}{2}(4+16) = \frac{20}{2} = 10$$

Example 4.30. Suppose $g(t) = 3\cos(2t) + 4\cos(2t - 30^\circ) + 5\sin(3t)$. Find P_g.

$5\cos(3t - 90^\circ)$

$$3\angle 0^\circ + 4\angle -30^\circ$$

$$\approx 6.77\angle -17.2^\circ$$

$$= \sqrt{2^2 + (3+2\sqrt{3})^2}\ \angle\theta$$

$$= \sqrt{25+12\sqrt{3}}\ \angle\theta$$

$\sqrt{4^2-2^2} = \sqrt{16-4} = 2\sqrt{3}$

$\sqrt{2^2 + (3+2\sqrt{3})^2}$

$$g(t) = \underbrace{\sqrt{25+12\sqrt{3}}}_{A_1}\cos(2t + \theta) + 5\cos(3t - 90^\circ)$$

"f_1" $\quad A_2 \quad$ "f_2" $\quad 0 < f_1 \neq f_2 > 0$

$$P_g = \frac{1}{2}|A_1|^2 + \frac{1}{2}|A_2|^2 = \frac{1}{2}\left(25+12\sqrt{3}+25\right) = 25+6\sqrt{3}$$

4.31. For ***periodic*** signal $g(t)$ with period T_0, there is also no need to carry out the limiting operation to find its (average) power P_g. We only need to find an average carried out over a single period:

$$P_g = \frac{1}{T_0}\int_{T_0} |g(t)|^2 dt.$$

Example 4.32.

4.33. When the Fourier series expansion (to be reviewed in Section 4.3) of the signal is available, it is easy to calculate its power:

(a) When the corresponding Fourier series expansion $g(t) = \sum\limits_{k=-\infty}^{\infty} c_k e^{j2\pi k f_0 t}$ is known,

$$P_g = \sum_{k=-\infty}^{\infty} |c_k|^2.$$

(b) When the signal $g(t)$ is real-valued and its (compact) trigonometric Fourier series expansion $g(t) = c_0 + 2\sum\limits_{k=1}^{\infty} |c_k| \cos(2\pi k f_0 t + \angle\phi_k)$ is known,

$$P_g = c_0^2 + 2\sum_{k=1}^{\infty} |c_k|^2.$$

Definition 4.34. Based on Definitions 4.13 and 4.15, we can define three distinct classes of signals:

$0 < E_g < \infty$

(a) If E_g is finite and nonzero, g is referred to as an ***energy signal***.

$0 < P_g < \infty$

(b) If P_g is finite and nonzero, g is referred to as a ***power signal***.

(c) Some signals[17] are neither energy nor power signals.

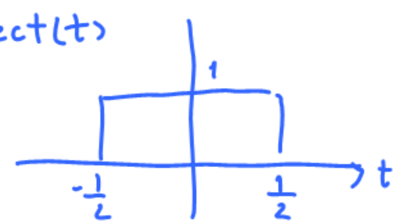
so a power signal cannot be an energy signal

- Note that the power signal has infinite energy and an energy signal has zero average power; thus the two categories are disjoint.

So, an energy signal cannot be a power signal

Example 4.35. Rectangular pulse

$g(t) = \mathrm{rect}(t)$



$$E_g = \int_{-\infty}^{\infty} |g(t)|^2 dt = \int_{-1/2}^{1/2} 1^2 dt$$

$$= 1^2 \times 1 = 1$$

$$1^2 t \Big|_{-1/2}^{1/2} = 1^2\underbrace{\left(\tfrac{1}{2} - \left(-\tfrac{1}{2}\right)\right)}_{1}$$

$0 < E_g < \infty$ $\Rightarrow$ g is an energy signal $\Rightarrow$ $P_g = 0$

$\Downarrow$

g is not a power signal

[17] Consider $g(t) = t^{-1/4} 1_{[t_0,\infty)}(t)$, with $t_0 > 0$.

$0 < E_g < \infty \Rightarrow g$ is an energy signal $\Rightarrow P_g = 0$

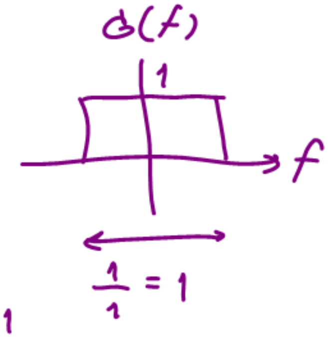
$\Downarrow$

g is not a power signal

Example 4.36. Sinc pulse

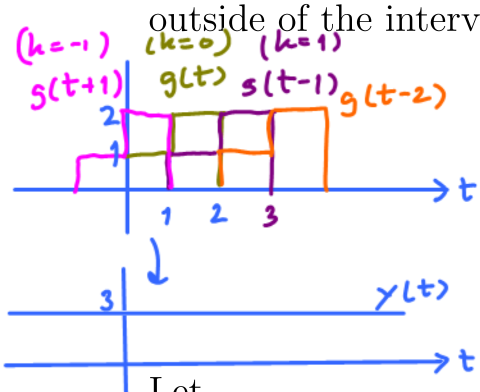
$g(t) = \text{sinc}(\pi t)$

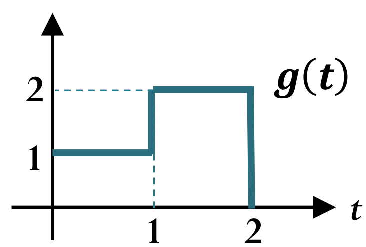
$\xrightarrow{\mathcal{F}}$



$$E_g = \int_{-\infty}^{\infty} |g(t)|^2 dt = \int_{-\infty}^{\infty} |\text{sinc}(\pi t)|^2 dt \overset{\text{Parseval}}{=} \int_{-\infty}^{\infty} |G(f)|^2 df = 1^2 \times 1 = 1$$

Example 4.37 (M2018)**.** Consider a signal $g(t)$ below. Note that $g(t) = 0$ outside of the interval $[0, 2]$.

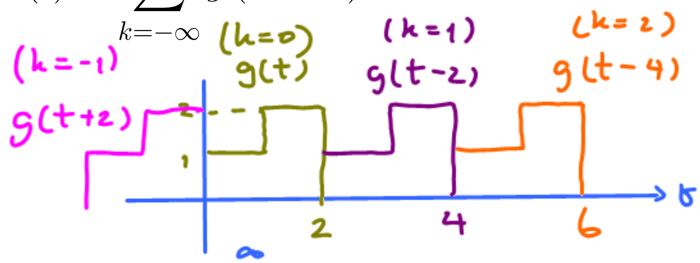




Let

$$E_y = \int_{-\infty}^{\infty} 3^2 dt = \infty$$

$$y(t) = \sum_{k=-\infty}^{\infty} g(t-k) \quad \text{and} \quad z(t) = \sum_{k=-\infty}^{\infty} g(t-2k).$$



$$E_z = \int_{-\infty}^{\infty} |z(t)|^2 dt = \infty$$

Calculate the following quantities:

(a) energy $E_g \equiv \int_{-\infty}^{\infty} |g(t)|^2 dt$

$= 1 \times 1 + 1 \times 4$

$= 5$

(b) average power $P_g \equiv \langle |g(t)|^2 \rangle$

$$= \lim_{T \to \infty} \frac{\int_{-T}^{T} |g(t)|^2 dt}{2T} = \frac{E_g}{\infty} = \frac{5}{\infty} = 0$$

(c) $\langle g(t) \rangle = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} g(t)\, dt = \lim_{T \to \infty} \frac{\text{area under } g(t) \ \cancel{\text{btw } -T \text{ to } T}}{2T} = \frac{3}{\infty} = 0$

when T is large enough

As $T \to \infty$, this term eventually converges to area under "g" $= (1 \times 1) + (1 \times 2) = 3$

(d) average power P_y $\quad P_y \equiv \langle |y(t)|^2 \rangle = \langle 3^2 \rangle = \langle 9 \rangle = 9$

$\langle y(t) \rangle = \langle c \rangle = c = 3$

(e) average power P_z $\quad P_z \equiv \frac{1}{T_0} \int_{T_0} |z(t)|^2 dt = \frac{1}{2} \int_0^2 |g(t)|^2 dt = \frac{1}{2} \times 5 = \frac{5}{2}$

$\langle z(t) \rangle = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} z(t) dt = \frac{1+2}{2} = \frac{3}{2}$

Note that $z(t)$ is periodic.
We have additional area of size 3 every 2 time units.

The table below summarizes, for each signal, its (i) time average (ii) (total) energy, (iii) (average) power, and indication (by putting a Y (for a yes) or an N (for a no)) in part (iv) whether it is an energy signal and in part (v) whether it is a power signal.

		$g(t)$	$y(t)$	$z(t)$
(i)	$\langle \cdot \rangle$	0	3	3/2
(ii)	(Total) Energy	5	∞	∞
(iii)	(Average) Power	0	9	5/2
(iv)	Energy Signal?	Y	N	N
(v)	Power Signal?	N	Y	Y

$0 < E < \infty$

$0 < P < \infty$

Example 4.38. For $\alpha > 0$, $g(t) = Ae^{-\alpha t} 1_{[0,\infty)}(t)$ is an energy signal with $E_g = |A|^2/2\alpha$.

Assume $c \neq 0$

$0 < P_g < \infty$

Example 4.39. The rotating phasor signal $g(t) = ce^{j(2\pi f_0 t + \theta)}$ is a power signal with $P_g = |c|^2$.

Assume $A \neq 0$, $f_0 \neq 0$

Example 4.40. The sinusoidal signal $g(t) = A\cos(2\pi f_0 t + \theta)$ is a power signal with $P_g = |A|^2/2$.

These are power signals.

$\Rightarrow$ They are not energy signals

$\Rightarrow E_g = \infty$

4.41. Consider the transmitted signal

$$x(t) = m(t)\cos(2\pi f_c t + \theta)$$

in DSB-SC modulation. Suppose $M(f - f_c)$ and $M(f + f_c)$ do not overlap (in the frequency domain).

(a) If $m(t)$ is a power signal with power P_m, then the average transmitted power is

$$P_x = \frac{1}{2}P_m.$$

"Proof"

$P_m = \sum_k |a_k|^2$

$P_x = 2\left(\sum_k \left|\frac{1}{2}a_k\right|^2\right) = \frac{1}{2}\underbrace{\left(\sum_k |a_k|^2\right)}_{P_m}$

- Q: Why is the power (or energy) reduced?

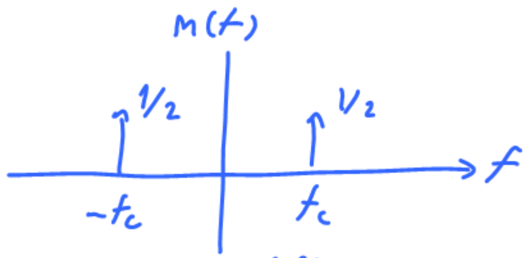
- Remark: When $x(t) = \sqrt{2}m(t)\cos(2\pi f_c t + \theta)$ (with no overlapping between $M(f - f_c)$ and $M(f + f_c)$), we have $P_x = P_m$.

(b) If $m(t)$ is an energy signal with energy E_m, then the transmitted energy is

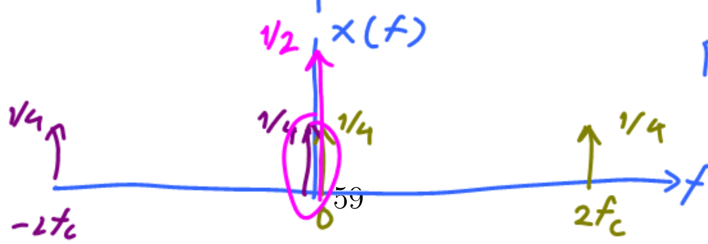
$$E_x = \frac{1}{2}E_m.$$

Example 4.42. Suppose $m(t) = \cos(2\pi f_c t)$. Find the average power in $x(t) = m(t)\cos(2\pi f_c t)$.

$P_m = \frac{1}{2}$



$P_m = \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 = \frac{1}{2}$



$P_x = \left(\frac{1}{4}\right)^2 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{4}\right)^2 = \frac{3}{8}$